ISRAEL JOURNAL OF MATHEMATICS 103 (1998), 289–318

A GEOMETRICAL APPROACH TO THE K-DIVISIBILITY PROBLEM

BY

PAVEL SHVARTSMAN*

Department of Mathematics, Technion—Israel Institute of Technology
32000 Haifa, Israel
e-mail: pshv@techunix.technion.ac.il

ABSTRACT

We give a geometrical interpretation of the Brudnyi-Krugljak K-divisibility theorem — one of the fundamental results of modern interpolation theory of Banach spaces. We show that this result is closely connected with a curious intersection theorem which can be formulated in the spirit of Helly's classical theorem. Let B_0, B_1 be two closed convex balanced subsets of a Banach space X. We prove that under a wide range of various conditions the family of sets $\mathcal{B} = \{B = sB_0 + tB_1 + c;\ \ s,t \in \mathbf{R}, c \in X\}$ possesses the following intersection property:

Let $\mathcal{B}'$ be a subfamily of $\mathcal{B}$ such that every two sets from $\mathcal{B}'$ have a common point. Then $\bigcap_{B\in\mathcal{B}'} \gamma \circ B \neq \emptyset$, where $\gamma > 0$ is an absolute constant ($\gamma \le 7 + 4\sqrt{2}$) and the symbol $\gamma \circ B$ denotes a dilation of B with respect to its center by a factor of γ.

As a consequence we obtain a generalization of the K-divisibility theorem for sums of two elements.

1. Introduction and results

In this paper we present a geometrical approach to the K-divisibility theorem of Yu. Brudnyi and N. Krugljak [BK1, BK2]. This theorem at the present time is one of the fundamental results of modern interpolation theory of Banach spaces.

* Supported by the Center for Absorption in Science, Israel Ministry of Immigrant Absorption and by grant No. 95-00225 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.
Received December 20, 1995 and in revised form February 12, 1997

It has many interesting applications both in interpolation theory and in various branches of analysis (see e.g. [BK2, Ch. 3,4], [C], [N1], [N2], [K] [M], [BS]).

Here we will deal with a particular case of the theorem relating to decompositions of an element of a Banach space into a sum of *two* terms. Let us recall this result.

Let $\vec{X} = (X_0, X_1)$ be a *compatible* pair of Banach spaces equipped with the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ (for this and the following definitions see, e.g., [BL]). We let $K(\cdot; x : \vec{X})$ denote the K-functional of an element $x \in X_0 + X_1$ with respect to the pair $\vec{X}$, i.e.,

$$K(t; x\colon \vec{X}) = \inf\{\|x_0\|_0 + t\|x_1\|_1 : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

THEOREM ((K-divisibility, [BK1, BK2])): *Suppose that*

$$K(t; x : \vec{X}) \le \varphi_1(t) + \varphi_2(t) \quad \textit{for all } t > 0$$

where each $\varphi_i(t)$ is a positive concave function on $(0,\infty)$. Then there exist elements $x_1, x_2 \in X_0 + X_2$ such that

$$x = x_1 + x_2,$$

$$K(t; x : \vec{X}) \le \gamma\varphi_i(t), \quad i = 1, 2, \quad t > 0.$$

Here γ is an absolute constant.

Brudnyi and Krugljak obtained that the constant γ satisfies $1 \le \gamma \le 14$. Then M. Cwikel [C] showed that $\gamma \le 8 + \varepsilon$, where $\varepsilon > 0$ is arbitrary (see also [BSh]). The best estimate for γ known at present is $\gamma \le 3 + 2\sqrt{2} + \varepsilon$, which was obtained by M. Cwikel, B. Jawerth and M. Milman [CJM].

We will be interested in a geometrical interpretation of the K-divisibility theorem. We will connect this assertion with a purely geometrical theorem which is very close in spirit to the classical Helly theorem [DGK]. As a corollary of the main result we obtain a generalization of the K-divisibility property (Theorem 1.2).

For the formulation of the result we let $B_i = \{x \in X_i : \|x\|_i \le 1\}$ denote the unit ball of the space X_i for $i = 0, 1$ and we introduce a family $\mathcal{B} = \mathcal{B}(\vec{X})$ of sets

$$\mathcal{B}(\vec{X}) = \{B = s\overline{B}_0 + t\overline{B}_1 + c; s, t \in \mathbf{R} \cup \{\pm\infty\}, c \in X_0 + X_1\}. \tag{1.1}$$

Here and below $\overline{B}$ (or $\mathrm{cl}_\Sigma(B)$) denotes the closure of a set B in the topology of X_0+X_1 (in case $s=\pm\infty$ we set $s\overline{B}_i=\overline{X}_i, i=0,1$). If B is centrally symmetric and λ is non-negative, then we let $\lambda\circ B$ denote a dilation of B with respect to its center by a factor of λ.

Let us consider the following main

Intersection Problem: Let $\mathcal{B}'$ be a subfamily of $\mathcal{B}(\vec{X})$ such that every two sets from $\mathcal{B}'$ have a common point. Does there exist a constant γ such that

$$\bigcap_{B\in\mathcal{B}'}\gamma\circ B\neq\emptyset?$$

Our first main result shows that the problem formulated above has a positive answer, at least in the case of *finite* families $\mathcal{B}'\subset\mathcal{B}(\vec{X})$:

THEOREM 1.1: *Let $\mathcal{B}'$ be a finite subfamily of $\mathcal{B}(\vec{X})$ such that every two sets of $\mathcal{B}'$ have a non-empty intersection. Then*

$$\bigcap_{B\in\mathcal{B}'}\gamma\circ B\neq\emptyset.$$

Here γ is an absolute constant which can be assumed to be of the form

$$\gamma=7+4\sqrt{2}+\varepsilon, \tag{1.2}$$

where ε is an arbitrary positive number.

We shall also present results for *infinite* families $\mathcal{B}'\subset\mathcal{B}(\vec{X})$ in Theorem 1.3 below. However, let first us formulate a corollary of Theorem 1.3 which generalizes the K-divisibility property. (I am very obliged to N. Krugljak who kindly drew my attention to this corollary.)

THEOREM 1.2: *Let $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ be a subset of X_0+X_1 and let $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of non-negative concave functions defined on $(0,\infty)$. Suppose that at least one of the following conditions holds:*

(i) *$\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is a weakly locally precompact set of X_0+X_1;*

(ii) *the imbedding $X_i\hookrightarrow X_0+X_1$, $i=0,1$, is either weakly compact or closed.*

Assume also that for any α and $\beta\in\mathcal{A}$ and $t>0$

$$K(t;x_\alpha-x_\beta:\vec{X})\leq\varphi_\alpha(t)+\varphi_\beta(t). \tag{1.3}$$

Then there exists an element $\bar{x} \in X_0 + X_1$ *such that for any* $\alpha \in \mathcal{A}$ *and* $t > 0$

$$K(t; x_\alpha - \overline{x} : \vec{X}) \leq \lambda \varphi_\alpha(t).$$

Here λ *can be of the form* $\lambda = 2(7 + 4\sqrt{2}) + \varepsilon$, *where* $\varepsilon > 0$ *is arbitrary.*

Note that in the case $\mathcal{A} = \{0,1\}, x_0 = 0, x_1 = x$ this assertion gives the K-divisibility theorem.

We now consider the Intersection Problem for infinite family of sets. The following theorem shows that for a rather large collection of Banach pairs $\vec{X}$ and families $\mathcal{B}' \subset \mathcal{B}(\vec{X})$ this problem has a positive answer. Let $c(\mathcal{B}')$ and $\mathcal{T}(\mathcal{B}')$ denote the set of centers and the set of the dilation coefficients of all the sets of $\mathcal{B}'$ respectively, i.e.

$$c(\mathcal{B}') = \{c \in X_0 + X_1 : \text{there exists } B = s\overline{B}_0 + t\overline{B}_1 + c \in \mathcal{B}'\}, \tag{1.4}$$

$$\mathcal{T}(\mathcal{B}') = \{(s,t) \in \mathbf{R}^2 : \text{ there exists } B = s\overline{B}_0 + t\overline{B}_1 + c \in \mathcal{B}'\}. \tag{1.5}$$

THEOREM 1.3: *Let* $\mathcal{B}'$ *be a subfamily of* $\mathcal{B}(\vec{X})$ *such that every two sets of* $\mathcal{B}'$ *have a non-empty intersection. Suppose also that at least one of the following conditions is fulfilled:*

(i) $\mathcal{T}(\mathcal{B}')$ *contains those of its limit points which lie on the coordinate axes;*

(ii) $c(\mathcal{B}')$ *is a weakly locally precompact subset of* $X_0 + X_1$;

(iii) *the imbedding* $X_i \hookrightarrow X_0 + X_1$, $i = 0, 1$ *is either weakly compact or closed*.*

Then

$$\bigcap_{B \in \mathcal{B}'} \gamma \circ B \neq \emptyset,$$

where γ *is the constant which appears in Theorem* 1.1.

Examples: Let us mention two other convenient conditions under which Theorem 1.3 can be applied:

(i)′ $\mathcal{T}(\mathcal{B}')$ is a closed subset of $\mathbf{R}^2$;

(ii)′ the linear span of $c(\mathcal{B}')$ is finite dimensional.

Obviously (i)′ implies (i) and (ii)′ implies (ii).

(iii)′ Let us also list some examples of *imbedded pairs* (i.e., $X_1 \hookrightarrow X_0$) of Banach spaces of smooth and/or integrable functions in which the corres-

* I.e., B_i is a weakly precompact subset of $X_0 + X_1$ or X_i is a closed linear subspace of $X_0 + X_1$.

ponding embedding operator is either compact or weakly compact. In particular, for the Banach couples $(C[0,1]^n, \mathrm{Lip}[0,1]^n)$, $(C^k[0,1]^n, C^m[0,1]^n)$, $k < m$, $(\mathrm{Lip}_\alpha[0,1]^n, \mathrm{Lip}_\beta[0,1]^n)$, $0 < \alpha < \beta \leq 1$, or more generally for the pairs $(L_r[0,1]^n, W_p^k[0,1]^n)$, $k/n > 1/p - 1/r$, $(L_r[0,1]^n, B_p^{\alpha q}[0,1]^n)$, $\alpha/n > 1/p - 1/r$, $1 \leq r, p \leq \infty$, this operator is *compact*. For the pairs $(l_p, l_q), 1 \leq q < p \leq \infty$ and $(L_p[0,1]^n, L_q[0,1]^n), 1 \leq p < q \leq \infty$ the imbedding operator is *weakly compact* and for the couples (l_∞, c_0) and $(\mathrm{Lip}[0,1]^n, C^1[0,1]^n)$ it is *closed*.

Imbedded pairs (X_0, X_1) in which at least one of the spaces X_0 or X_1 is *reflexive* provide further examples of weakly compactly imbedded couples. In fact, here, by the Banach–Alaoglu theorem, either B_1 is a weakly compact subset of X_1 or B_0 is a weakly compact subset of X_0 . Since $X_1 \hookrightarrow X_0$, it follows that in both cases B_1 is weakly precompact in the topology of X_0 and consequently the imbedding operator is weakly compact.

By Theorem 1.3 the answer to the Intersection Problem is affirmative for all the Banach couples listed in (iii)′ and for every family $\mathcal{B}'$ satisfying condition (i)′ or (ii)′. ∎

In connection with the statement of Theorem 1.3 the following question is quite natural: is the theorem valid without the condition (i)–(iii)? We do not know the answer to this question. Nevertheless, the method of proof permits us to give an "intrinsic" description of those Banach pairs for which the conclusion of the theorem does hold.

For the formulation of the result we need the following

Definition 1.4: Let X, Y be Banach spaces, $Y \hookrightarrow X$. We shall say that X is *strongly complete* with respect to Y, if there exists a constant λ, $\lambda \geq 1$ such that for any sequence $\{x_i\}_{i=1}^\infty \subset X$ the condition

$$\mathrm{dist}(x_i - x_j, B_Y) \to 0, \quad i, j \to \infty$$

implies the existence of an element $x \in X$ such that

$$\mathrm{dist}(x_i - x, \lambda B_Y) \to 0, \quad i \to \infty. \tag{1.6}$$

Here, B_Y is the unit ball of Y and

$$\mathrm{dist}(x, B_Y) = \inf_{y \in B_Y} \|x - y\|_X. \qquad ∎$$

Let us denote by $\lambda_X(Y)$ the smallest value of the constant λ for which the above property holds.

Examples: (a) If $Y = \{0\}$ is the null space, then strong completeness of X with respect to Y coincides with ordinary completeness of X.

(b) Let Y be a closed subspace of X with seminorm $\|\cdot\|_Y \equiv 0$ (i.e., $B_Y = Y$). Then the strong completeness of X with respect to Y is equivalent to the (ordinary) completeness of the quotient space X/Y. ∎

THEOREM 1.5:

(i) *Suppose that the space $\Sigma = X_0 + X_1$ is strongly complete both with respect to X_0 and with respect to X_1. If every two sets of a subfamily $\mathcal{B}' \in \mathcal{B}(\vec{X})$ have a common point, then $\bigcap\{\gamma \circ B : B \in \mathcal{B}'\} \neq \emptyset$ where*

$$\gamma \leq 3(7+4\sqrt{2})\max(\lambda_\Sigma(X_0), \lambda_\Sigma(X_1)). \tag{1.7}$$

(ii) *Conversely, assume that for some γ, $\gamma \geq 1$ the following condition holds: if $\mathcal{B}'$ is an arbitrary subfamily of $\mathcal{B}(\vec{X})$ such that every two sets of $\mathcal{B}'$ have non-empty intersection, then $\bigcap\{\gamma \circ B : B \in \mathcal{B}'\} \neq \emptyset$. Then $X_0 + X_1$ is strongly complete both with respect to X_0 and with respect to X_1. Moreover*

$$\lambda_\Sigma(X_i) \leq \tfrac{1}{2}\gamma, \quad i = 0, 1. \tag{1.8}$$

Before presenting the proofs of the results formulated above let us make some remarks.

Remark 1.6: This investigation was inspired by an interesting geometrical interpretation of the K-divisibility property, obtained by N. Krugljak [K1]:

Let U_1, U_2 be two arbitrary subsets of the positive octant $\{(s,t) : s > 0, t > 0\}$. Then

$$\bigcap_{(s,t)\in U_1+U_2} (sB_0 + tB_1) \subset \gamma\{\bigcap_{(s,t)\in U_1} (sB_0 + tB_1) + \bigcap_{(s,t)\in U_2} (sB_0 + tB_1)\}, \tag{1.9}$$

where γ is an absolute constant. Note that the converse imbedding is trivial and holds with $\gamma = 1$.

Let us show that the imbedding (1.9) is a corollary of Theorem 1.3. To prove that a point x belonging to the left-hand side of (1.9) belongs also to the right-hand side, it is sufficient to apply Theorem 1.3 to the family

$$\mathcal{B}' = \{B = s\overline{B}_0 + t\overline{B}_1, (s,t) \in U_1\} \cup \{B = s\overline{B}_0 + t\overline{B}_1 + x, (s,t) \in U_2\} \tag{1.10}$$

and to make use of the positivity of the coefficients s, t in (1.10). Since the set of centers $c(\mathcal{B}')$ consists of only two points, condition (ii) of Theorem 1.3 is obviously fulfilled in this case. ∎

Remark 1.7: Since the formulation of Theorem 1.1 is very similar to that of Helly's Theorem in one dimension, it is natural to ask the following question: Does there exist an analog of Helly's Theorem [DGK] (in the spirit of Theorem 1.1) for the family

$$\mathcal{B}(\vec{X}_n) = \Big\{B = \sum_{i=0}^{n} t_i \overline{B}_i; t_i \in \mathbf{R} \cup \{\pm\infty\}, i = 0, \ldots, n, c \in \sum_{i=0}^{n} X_i\Big\}$$

generated by an $(n+1)$-tuple of Banach spaces $\vec{X}_n$? In other words, does there exist a positive integer $m = m(n)$ and a constant $\gamma = \gamma(n)$ such that for any family $\mathcal{B}' \subset \mathcal{B}(\vec{X})$ in which every m sets have a common point, the intersection $\bigcap\{\gamma \circ B : B \in \mathcal{B}'\}$ is non-empty?

Example 5.1, given in Section 5, shows that the number $m(n)$ *cannot be finite* even if we restrict ourselves to finite-dimensional Banach triples (but of arbitrary dimension).

Remark 1.8: The constant γ from (1.2) can probably be improved considerably. In particular we can prove that for all *two-dimensional* Banach pairs, $\gamma \leq 2$. This estimate of γ is exact and is attained, for instance, for $B_0 = [-1,1]^2$ and $B_1 = \{(x,y) : y = x, -1 \leq x \leq 1\}$. The point of view developed here, of viewing the K-divisibility problem as an intersection theorem (cf. Remark 1.6), also provides an approach for calculating the K-divisibility constant. It can be shown that for all two-dimensional pairs this constant is less than or equal to $(\sqrt{2}+1)/(3-\sqrt{2}) \approx 1.52$. This value is attained, for example, when $B_0 = [-1,1]^2$ and $B_1 = \{(x,y) : y = \tan(\frac{\pi}{8})x, -1 \leq x \leq 1\}$. This fact, as well as several other results concerning the K-divisibility constant, will be presented in a forthcoming paper [S]. ∎

The paper is organized as follows. Section 2 contains definitions and the proof of the main Proposition 2.1 (which is actually is an extension of part (i) of Theorem 1.3). Section 3 is devoted to the proof of a series of auxiliary lemmas. Section 4 contains the proof of Theorem 1.3 and Theorem 1.5. Finally, in Section 5 we present Example 5.1, mentioned in Remark 1.7, and prove Theorem 1.2.

ACKNOWLEDGEMENT: I am deeply grateful to N. Krugljak and V. L. Dolnikov for very stimulating and helpful discussions and remarks. I should also like to thank Y. Benyamini, Yu. Brudnyi and M. Cwikel for valuable advice and help.

2. Auxiliary statements

Let us recall that the norm in the space $\Sigma = X_0 + X_1$ is defined by

$$\|x\|_\Sigma = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\} \tag{2.1}$$

and consequently, for any $\varepsilon > 0$,

$$\bar{B}_0 \cup \bar{B}_1 \subset B_\Sigma \subset (1+\varepsilon)(B_0 + B_1). \tag{2.2}$$

Here and below B_Σ denotes the unit ball of $X_0 + X_1$. Later on we shall also need the following elementary properties of subsets from $X_0 + X_1$. Namely, by (2.2), for any $S \subset X_0 + X_1$

$$\bar{S} = \bigcap_{\varepsilon>0}(S + \varepsilon B_\Sigma) = \bigcap_{\varepsilon>0}(S + \varepsilon(B_0 + B_1)). \tag{2.3}$$

Hence, in particular, it follows that for any set $B \in \mathcal{B}(\vec{X})$ and any $\varepsilon > 0$

$$B \subset \bar{B} \subset (1+\varepsilon) \circ B. \tag{2.4}$$

If $B = s\bar{B}_0 + t\bar{B}_1 + c$, then in the cases $s = 0$ or $t = 0$ property (2.4) is obvious. If s and t are both non-zero, then, by (2.3),

$$\bar{B} \subset s\bar{B}_0 + t\bar{B}_1 + c + \varepsilon \min(s,t)(B_0 + B_1) \subset (1+\varepsilon)(s\bar{B}_0 + t\bar{B}_1) + c = (1+\varepsilon) \circ B.$$

Let us remark also that for any sets $B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i,\ i = 1,2$

$$B^{(1)} \cap B^{(2)} \neq \emptyset \iff c_1 - c_2 \in (s_1 + s_2)\bar{B}_0 + (t_1 + t_2)\bar{B}_1. \tag{2.5}$$

Note at last that for any set $B = s\bar{B}_0 + t\bar{B}_1 + c \in \mathcal{B}(\vec{X})$

$$B = |s|\bar{B}_0 + |t|\bar{B}_1 + c.$$

Therefore, everywhere below, we can assume that $s, t \geq 0$, i.e.,

$$\mathcal{T}(\mathcal{B}') \subset \mathbf{R}_+^2 = \{(s,t) : s,t \geq 0\}$$

(see (1.5) for the definition of $\mathcal{T}(\mathcal{B}')$). To formulate the main result of the section we need the following notations.

Throughout the proof $\mathcal{B}'$ denotes a subfamily of $\mathcal{B}(\vec{X})$ such that *every two sets of $\mathcal{B}'$ have a common point*. For each subset $A \subset \mathbf{R}^2$, let s_A denote the point nearest to the origin among all limit points of A lying on the s-axis. Analogously t_A denotes the limit point on the t-axis which is nearest to the origin. For simplicity of notation we shall write $s(\mathcal{B}')$ and $t(\mathcal{B}')$ instead of $s_{\mathcal{T}(\mathcal{B}')}$ and $t_{\mathcal{T}(\mathcal{B}')}$.

PROPOSITION 2.1: *Suppose that the points $s(\mathcal{B}')$ and $t(\mathcal{B}')$ belong to $\mathcal{T}(\mathcal{B}')$. Then for some $\gamma > 0$*

$$\bigcap_{B \in \mathcal{B}'} \gamma \circ B \neq \emptyset.$$

Here γ is an absolute non-negative constant, $\gamma \leq (7 + 4\sqrt{2}) + \varepsilon$, where $\varepsilon > 0$ is an arbitrary positive number.

Proof: The proposition is based on two auxiliary lemmas.

LEMMA 2.2: *Let A be a subset of $\mathbf{R}^2_+$ such that s_A and t_A belong to A. Then for arbitrary $r > 1$ and $\varepsilon > 0$ there exists a countable or finite subset*

$$U = \{u_i = (s_i, t_i) : i \in I\} \subset A$$

such that:

(a) *for any $i \in I$*

$$\sum_{k<i} s_k \leq (1+\varepsilon)\frac{r}{r-1}s_i, \quad \sum_{k>i} t_k \leq (1+\varepsilon)\frac{r}{r-1}t_i; \tag{2.6}$$

(b) *for any $z = (s, t) \in A$ there exists a point $u_i = (s_i, t_i) \in U$ such that*

$$s_i \leq (r+\varepsilon)s, \quad t_i \leq (r+\varepsilon)t.$$

Proof: Let us set

$$G = \{z = (x, y) \in \mathrm{cl}(A) : \{(s, t) : s \leq x, t \leq y\} \cap \mathrm{cl}(A) = z\} \tag{2.7}$$

where $\mathrm{cl}(A)$ denotes the closure of A. It is easy to check that the set $G \neq \emptyset$ and possesses the following properties:

(1) if $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in G$, $z_1 \neq z_2$, then $x_1 \neq x_2, y_1 \neq y_2$. Moreover, if $x_1 < x_2$, then $y_1 < y_2$;

(2) suppose that a sequence $\{z_i = (x_i, y_i)\}_{i=1}^{\infty} \subset G$ converges to a point $z = (x, y)$ and $x_i \geq x$, $y_i \leq y$ for all $i = 1, 2, \dots$; then $z \in G$.

In fact G is a graph of some non-negative monotone decreasing and continuous from the right function, defined on a subset of $\mathbf{R}_+$.

Let us construct the set U, satisfying the lemma's conditions. To this end we let J denote a set of integral indexes j such that

$$G_j = G \cap \{(s,t) \in \mathbf{R}_+^2 : r^j < t < r^{j+1}\} \neq \emptyset.$$

Let us observe that if A has limit points on the s-axis then, by definition (2.7), the point s_A belongs to G. In this case we set $G_{-\infty} = \{s_A\}$.

Given $j \in J$ let us define numbers x_j, y_j by the formulas

$$x_j = \inf\{s : (s,t) \in G_j\}, \quad y_j = \sup\{t : (s,t) \in G_j\}. \tag{2.8}$$

By properties (1), (2) of the set G, it follows that the point $z_j = (x_j, y_j)$ belongs to G_j. Set now

$$H = \{z_j \colon j \in J\}$$

and denote by I a family of integral indexes such that the set

$$H_i = H \cap \{(s,t) \in \mathbf{R}_+^2 : r^i \leq s < r^{i+1}\} \tag{2.9}$$

is non-empty. Note that if A has limit points on the t-axis, then the point t_A belongs to H. In this case we put $H_{-\infty} = \{t_A\}$.

Let us set now

$$\tilde{s}_i = \sup\{s : (s,t) \in H_i\}, \quad \tilde{t}_i = \inf\{t : (s,t) \in H_i\}. \tag{2.10}$$

By discreteness of H_i and property (1) of the set G, the point $\tilde{u}_i = (\tilde{s}_i, \tilde{t}_i)$ belongs to H_i. Let us denote by $\tilde{U}$ the set

$$\tilde{U} = \{\tilde{u}_i : i \in I\}$$

and define the desired set U.

If the point $\tilde{u}_i$ lies on a coordinate axis (and consequently $\tilde{u}_i = s_A$ or t_A) then we set $u_i = \tilde{u}_i$. By property (1), the remaining points of $\tilde{U}$ have positive coordinates. Let $\tilde{u}_i = (\tilde{s}_i, \tilde{t}_i)$ be such a point. Since $\tilde{u}_i \in \mathrm{cl}(A)$ and $\tilde{s}_i, \tilde{t}_i > 0$, there exists a point $u_i = (s_i, t_i) \in A$ such that

$$(1-\delta)\tilde{s}_i \leq s_i \leq (1+\delta)\tilde{s}_i \tag{2.11}$$

and

$$(1-\delta)\tilde{t}_i \leq t_i \leq (1+\delta)\tilde{t}_i \tag{2.12}$$

where we set $\delta = \varepsilon/(\varepsilon+2)$.

Let us show that the set $U = \{u_i : i \in I\}$ satisfies properties (a) and (b) of the lemma. For the proof of (a) we note that, by (2.9),

$$\tilde{s}_k \leq r^{k-i+1}\tilde{s}_i \qquad \text{for } k < i.$$

Together with (2.11) this gives

$$s_k \leq (1+\delta)\tilde{s}_k \leq (1+\delta)r^{k-i+1}\tilde{s}_i \leq \frac{(1+\delta)}{(1-\delta)}r^{k-i+1}s_i.$$

Therefore

$$\sum_{k<i} s_i \leq \frac{(1+\delta)}{(1-\delta)}s_i\Big\{\sum_{k<i} r^{k-i+1}\Big\} \leq \frac{(1+\delta)}{(1-\delta)}\Big(\frac{r}{r-1}\Big)s_i.$$

Inequality (a) for the sequence $\{t_i : i \in I\}$ is proved analogously.

Let us prove (b). For each point $z = (s,t) \in A$ we define the point $z' = (x', y')$ by

$$x' = \min\{w : v \leq s, (v,w) \in \mathrm{cl}(A)\}, \quad y' = \min\{v : w = x', (v,w) \in \mathrm{cl}(A)\}.$$

Since $\mathrm{cl}(A)$ is closed, the point $z' = (x', y')$ belongs to $\mathrm{cl}(A)$. Moreover, by the definition of x', y' and (2.7), it follows that $z' \in G$ and

$$x' \leq s, \quad y' \leq t.$$

Let us now choose an index $j \in J$ such that $z' \in G_j$ and let $z'' = (x_j, y_j)$ denote the point defined by (2.8). Then, by the definition of G_i,

$$x_j \leq x' \leq s, \quad y_j \leq ry' \leq rt.$$

Let us next choose an index $i \in I$ such that $z'' \in H_i$. If $\tilde{u}_i = (\tilde{s}_i, \tilde{t}_i) \in \mathrm{cl}(A)$ denotes the point defined by formula (2.10), then

$$\tilde{s}_i \le rx'' = rx_j \le rs, \quad \tilde{t}_i \le y'' = y_j \le rt.$$

Finally, let $u_i = (s_i, t_i)$ be a point in U which satisfies inequalities (2.11) and (2.12). Then

$$s_i \le (1+\delta)\tilde{s}_i \le (1+\varepsilon)\tilde{s}_i \le rs,$$

$$t_i \le (1+\delta)\tilde{t}_i \le (1+\varepsilon)\tilde{t}_i \le rt,$$

and the lemma follows. ∎

Let further $B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i$, where $c_i \in X_0 + X_1$ and $s_i, t_i \ge 0$, $i \in I$, be a sequence of *pairwise intersecting* subsets of $\mathcal{B}(\vec{X})$.

LEMMA 2.3: *Suppose that for every $i \in I$*

$$\alpha_i = 2\sum_{k<i} s_k + s_i < \infty, \quad \beta_i = 2\sum_{k>i} t_k + t_i < \infty. \tag{2.13}$$

Then for any positive ε

$$\bigcap_{i\in I}\{(1+\varepsilon)\alpha_i\bar{B}_0 + (1+\varepsilon)\beta_i\bar{B}_1 + c_i\} \ne \emptyset.$$

Proof: Without loss of generality, we can choose the index set I to consist of a (finite or infinite) family of consecutive integers. Since $B^{(i)} \cap B^{(i+1)} \ne \emptyset$ for any $i, i+1 \in I$, by (2.5) there exist elements $x_i \in (s_i + s_{i+1})\bar{B}_0$ and $y_i \in (t_i + t_{i+1})\bar{B}_1$ such that

$$c_{i+1} - c_i = -x_i + y_i. \tag{2.14}$$

Let δ be an arbitrary positive number and $\lambda_i = \delta \cdot 2^{-|i|}$. Then, by (2.3),

$$\|x_i\|_0 \le (s_i + s_{i+1}) + (\lambda_i + \lambda_{i+1}), \tag{2.15}$$

$$\|y_i\|_1 \le (t_i + t_{i+1}) + (\lambda_i + \lambda_{i+1}).$$

In addition, by (2.14),

$$c_i + \sum_{i\le m<k} y_m = c_k + \sum_{i\le m<k} x_m. \tag{2.16}$$

Let us set now

$$\bar{c} = c_i + \sum_{m<i} x_m + \sum_{m\geq i} y_m. \tag{2.17}$$

By (2.13) and (2.15), both series in (2.17) are absolutely summable. Moreover, according to (2.16), the point $\bar{c}$ does not depend on i. Let us show that

$$\bar{c} \in (1+\varepsilon)\alpha_i \bar{B}_0 + (1+\varepsilon)\beta_i \bar{B}_1 + c_i, \quad i \in I.$$

We have

$$\begin{aligned}\|\sum_{m<i} x_m\|_0 &\leq \sum_{m<i} \|x_m\|_0 \leq \sum_{m<i}(s_m + s_{m+1} + \lambda_m + \lambda_{m+1}) \\ &\leq 2\sum_{m<i} s_m + s_i + 2\sum_{m\in I} \lambda_m \leq \alpha_i + 2\delta \sum_{m\in Z} 2^{-m} = \alpha_i + 6\delta.\end{aligned}$$

In the same way we obtain the estimate

$$\Big\|\sum_{m\geq i} y_m\Big\|_1 \leq \beta_i + 6\delta.$$

From this and (2.17) it follows that

$$\bar{c} \in \alpha_i B_0 + \beta_i B_1 + c_i + 6\delta(B_0 + B_1).$$

By the arbitrariness of δ, (2.3) and (2.4), we obtain the desired property

$$\bar{c} \in \mathrm{cl}_\Sigma(\alpha_i B_0 + \beta_i B_1 + c_i) \subset (1+\varepsilon)\alpha_i \bar{B}_0 + (1+\varepsilon)\beta_i \bar{B}_1 + c_i,$$

which proves the lemma. ∎

We are now in a position to complete the proof of Proposition 2.1. Let

$$A = \mathcal{T}(\mathcal{B}').$$

By the assumption of the proposition, the points $s_A = s(\mathcal{B}')$ and $t_A = t(\mathcal{B}')$ belong to A. Applying now to the set A the result of Lemma 2.2, we obtain that for given $r > 1$ and $\varepsilon > 0$ there exists a sequence $\{(s_i, t_i)\}_{i\in I} \subset A$ which satisfies properties (a), (b) of the lemma. Let us now set

$$\alpha_i = 2\sum_{k<i} s_k + s_i, \quad \beta_i = 2\sum_{k>i} t_k + t_i, \quad i \in I.$$

Then, by inequality (2.6),

$$\alpha_i \le \Big(2(1+\varepsilon)\Big(\frac{r}{r-1}\Big)+1\Big)s_i \tag{2.18}$$

and

$$\beta_i \le \Big(2(1+\varepsilon)\Big(\frac{r}{r-1}\Big)+1\Big)t_i. \tag{2.19}$$

Thus the sequence of sets

$$B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i, \quad i \in I$$

satisfies the conditions of Lemma 2.3 and consequently there exists an element $\bar{c} \in X_0 + X_1$ such that

$$\bar{c} \in (1+\varepsilon)\alpha_i\bar{B}_0 + (1+\varepsilon)\beta_i\bar{B}_1 + c_i.$$

From this and inequalities (2.18) and (2.19) we have

$$\bar{c} \in (1+\varepsilon)\Big(2(1+\varepsilon)\big(\frac{r}{r-1}\big)+1\Big)(s_i\bar{B}_0 + t_i\bar{B}_1) + c_i = \lambda \circ B^{(i)} \tag{2.20}$$

where

$$\lambda = (1+\varepsilon)\Big(2(1+\varepsilon)\big(\frac{r}{r-1}\big)+1\Big).$$

Next, by condition (b) of Lemma 2.2, we see that for every set

$$B = s\bar{B}_0 + t\bar{B}_1 + c \in \mathcal{B}'$$

there exists a pair (s_i, t_i) such that

$$s_i \le (r+\varepsilon)s, \quad t_i \le (r+\varepsilon)t.$$

Since $B^{(i)} \cap B \ne \emptyset$, by (2.5) it follows that

$$\bar{c} - c = (\bar{c} - c_i) + (c_i - c)$$

$$\in \lambda\{s_i\bar{B}_0 + t_i\bar{B}_1\} + (r+1+\varepsilon)\{s\bar{B}_0 + t\bar{B}_1\} \subset \gamma(r,\varepsilon)\{s\bar{B}_0 + t\bar{B}_1\},$$

where $\gamma(r,\varepsilon) = (\lambda(r+\varepsilon) + r + 1 + \varepsilon)$. Setting $r_0 = 1 + 1/\sqrt{2}$ we have

$$\gamma = \gamma(r_0, \varepsilon) = 7 + 4\sqrt{2} + O(\varepsilon).$$

Thus

$$\bar{c} - c \in \gamma(s\bar{B}_0 + t\bar{B}_1).$$

This means that $\bar{c} \in \gamma \circ B$ and Proposition 2.1 follows. ∎

3. Main lemmas

Suppose that there exists a sequence of sets $\{B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i,\, i = 1, 2, \ldots\} \subset \mathcal{B}'$ such that

$$\lim_{i\to\infty} s_i = u, \quad \lim_{i\to\infty} t_i = 0. \tag{3.1}$$

In this section we present a series of sufficient conditions under which the family $\mathcal{B}'$ possesses the following

Property 3.1: For any $\varepsilon > 0$ there exists an element $\bar{c} \in X_0 + X_1$ such that, for every $B \in \mathcal{B}'$,

$$((1+\varepsilon)u\bar{B}_0 + \bar{c}) \cap (1+\varepsilon) \circ B \neq \emptyset.$$

A sufficient condition for $\mathcal{B}'$ to have this property is given by the following

LEMMA 3.2: *If $\{c_i\}_{i=1}^{\infty}$ is a weakly convergent sequence in $X_0 + X_1$, then $\mathcal{B}'$ possesses Property* 3.1.

Proof: Since $B^{(i)} \cap B \neq \emptyset$ for all $i = 1, 2\ldots$ and every $B \in \mathcal{B}'$,

$$c_i \in B + uB_0 + \max(s_i - u, 0)\bar{B}_0 + t_i\bar{B}_1 \subset B + u\bar{B}_0 + v_iB_\Sigma$$

where we put

$$v_i = \max(s_i - u, 0) + t_i.$$

Note that by (3.1), v_i tends to 0 as $i \to \infty$. Let $\bar{c}$ denote the *weak limit* of the sequence $\{c_i\}$. Then for any functional $f \in \Sigma^*$

$$f(c_i) \leq \delta(f, B + u\bar{B}_0) + v_i\|f\|^*,$$

where $\delta(f, A) = \sup\{f(x) : x \in A\}$ denotes the support function of a set $A \subset \Sigma$. Since $f(c_i) \to f(\bar{c})$ and $v_i \to 0$ as $i \to \infty$, we obtain

$$f(\bar{c}) \leq \delta(f, B + u\bar{B}_0).$$

Hence, by the Hahn–Banach theorem

$$\bar{c} \in \mathrm{cl}_\Sigma(B + u\bar{B}_0),$$

which together with (2.3) gives

$$\bar{c} \in (1+\varepsilon) \circ (B + u\bar{B}_0) = (1+\varepsilon) \circ B + (1+\varepsilon)u\bar{B}_0.$$

Lemma 3.2 is proved. ∎

Before we formulate the following result let us remark that by the property $B^{(i)} \cap B^{(j)} \neq \emptyset$ and (2.5), we have

$$c_i - c_j \in (s_i + s_j)\bar{B}_0 + (t_i + t_j)\bar{B}_1 \tag{3.2}$$

for every $i, j \in \mathbf{N}$.

LEMMA 3.3: *If the set $c(\mathcal{B}')$ (see (1.4)) is weakly locally precompact in $X_0 + X_1$, then $\mathcal{B}'$ possesses Property* 3.1.

Proof: By (3.2),

$$c_i - c_j \in (s_i + s_j)\bar{B}_0 + (t_i + t_j)\bar{B}_1 \subset (s_i + s_j + t_i + t_j)B_\Sigma \tag{3.3}$$

for any $i, j \in \mathbf{N}$. In addition, by (3.1) the sequences $\{s_i\}$ and $\{t_i\}$ are bounded. From this and (3.3) it follows that the sequence $\{c_i\}$ is bounded in $X_0 + X_1$. Since $c(\mathcal{B}')$ is weakly locally precompact and $\{c_i\} \subset c(\mathcal{B}')$, there exists a weakly convergent subsequence $\{c_{i_k}\} \subset \{c_i\}$. It remains to apply to the sequence $\{c_{i_k}\}$ the result of Lemma 3.2 and the desired Property 3.1 follows. ∎

LEMMA 3.4: *If 0 is a limit point of the set $\mathcal{T}(\mathcal{B}')$, then for arbitrary $\varepsilon > 0$*

$$\bigcap_{B \in \mathcal{B}'} (1+\varepsilon) \circ B \neq \emptyset.$$

Proof: By the assumption, there exists a sequence of sets $\{B^{(i)}\}_{i=1}^{\infty} \subset \mathcal{B}'$ for which (3.1) holds with $u = 0$. Then, by (3.3), $\{c_i\}$ is a fundamental sequence in $X_0 + X_1$. In particular, $\{c_i\}$ is a weakly convergent sequence. Therefore, by Lemma 3.2,

$$((1+\varepsilon)u\bar{B}_0 + \bar{c}) \cap (1+\varepsilon) \circ B = \{\bar{c}\} \cap (1+\varepsilon) \circ B \neq \emptyset$$

for some element $\bar{c} \in X_0 + X_1$ and for every $B \in \mathcal{B}'$. ∎

LEMMA 3.5: *Let B_0 be a weakly precompact subset of $X_0 + X_1$. Then $\mathcal{B}'$ possesses Property* 3.1.

Proof: Let us show that the sequence $\{c_i\}$ has a *weakly convergent* subsequence $\{c_{i_k}\}$.

Setting

$$\tilde{s_i} = \max\{s_i - u, 0\},$$

by (3.2), we have

$$c_i - c_j \in (s_i + s_j)\bar{B}_0 + (t_i + t_j)\bar{B}_1 \subset 2u\bar{B}_0 + (\tilde{s}_i + \tilde{s}_j)\bar{B}_0 + (t_i + t_j)\bar{B}_1$$

$$\subset 2u\bar{B}_0 + (\tilde{s_i} + t_i + \tilde{s_j} + t_j)B_\Sigma.$$

Hence it follows that there exist elements $a_{ij}, b_{ij} \in X_0 + X_1$ such that

$$c_i - c_j = a_{ij} + b_{ij} \tag{3.4}$$

and

$$\{a_{ij}\} \subset 2u\bar{B}_0, \quad \{b_{ij}\} \subset (\varepsilon_i + \varepsilon_j)B_\Sigma \tag{3.5}$$

where $\varepsilon_i = \tilde{s_j} + t_j$. Note that by (3.1), ε_i converges to 0 as $i \to \infty$.

Let us now obtain a subsequence $\{a_{\alpha_i,\beta_j}\}$ of the (double) sequence $\{a_{ij}\}$ which has the additional property that for each fixed i, $\{a_{\alpha_i,\beta_j}\}$ is weakly convergent. We do this by the standard diagonalization procedure. (I.e., since B_0 is weakly precompact by (3.5), the set $\{a_{ij}\}$ is also weakly precompact. So $\{a_{1j}\}_{j=1}^{\infty}$ contains a weakly convergent subsequence $\{a_{1j_k}\}_{k=1}^{\infty}$. Then the sequence $\{a_{j_2j_k}\}_{k=2}^{\infty}$ also contains a weakly convergent subsequence, and we repeat this procedure for each subsequent value of k.)

From here on we can denote the new sequence $\{a_{\alpha_i,\beta_j}\}$ by $\{a_{ij}\}$. Observe that it also satisfies (3.4) for all $i, j \in \mathbf{N}$, $i < j$ and for each fixed i the (new) sequence $\{a_{ij}\}$ converges weakly as $j \to \infty$. We let a_i denote the weak limit of $\{a_{ij}\}$. Then, by (3.5), $\{a_i\} \subset 2u\bar{B}_0$ and consequently $\{a_i\}$ is a weakly precompact sequence.

Let us show now that the sequence

$$g_i = c_i - a_i$$

strongly (i.e., in the norm of $X_0 + X_1$) converges. Let k be a positive integer, $k \geq \max(i, j)$. Then, by (3.4),

$$g_i - g_j = (c_i - a_i) - (c_j - a_j) = (c_i - c_k - a_i) - (c_j - c_k - a_j)$$

$$= (a_{ik} - a_i) - (a_{jk} - a_j) + (b_{ik} - b_{jk}).$$

Let f be an arbitrary functional in Σ^* of the norm $\|f\|^* \le 1$. Then, by (3.5),

$$\begin{aligned} |f(g_i - g_j)| &\le |f(a_{ik} - a_i)| + |f(a_{jk} - a_j)| + |f(b_{ik})| + |f(b_{jk})| \\ &\le |f(a_{ik} - a_i)| + |f(a_{jk} - a_j)| + \varepsilon_i + \varepsilon_j + 2\varepsilon_k. \end{aligned}$$

Since $\varepsilon_k \to 0$ and $a_{ik} - a_i \to 0$ weakly as $k \to \infty$, we have

$$|f(g_i - g_j)| \le \varepsilon_i + \varepsilon_j.$$

By the arbitrariness of f, it means that

$$\|g_i - g_j\|_\Sigma \le \varepsilon_i + \varepsilon_j.$$

Thus $\{g_i\}$ is a fundamental and consequently a (strongly) convergent sequence.

Since $\{c_i = g_i + a_i\}$ is the sum of a strongly convergent sequence and a weakly precompact sequence, $\{c_i\}$ is also a weakly precompact sequence. In particular, $\{c_i\}$ contains a weakly convergent subsequence $\{c_{i_k}\}$.

It remains to apply to $\{c_{i_k}\}$ the result of Lemma 3.2 and the proof is complete. ∎

LEMMA 3.6: *Suppose that X_0 is a closed subspace of $X_0 + X_1$. Then there exists an element $a_{X_0} \in X_0 + X_1$ such that for every set $B = s\bar{B}_0 + t\bar{B}_1 + c \in \mathcal{B}'$,*

$$\operatorname{dist}_\Sigma(c - a_{X_0}, X_0) \le t.$$

Proof: Let Σ/X_0 denote the quotient space of $\Sigma = X_0 + X_1$ with respect to the space X_0 and let $\pi : \Sigma \to \Sigma/X_0$ be the corresponding projection operator. The statement of the lemma is equivalent to the existence of an element $z \in \Sigma/X_0$ such that for every $B = s\bar{B}_0 + t\bar{B}_1 + c \in \mathcal{B}'$,

$$\|z - \pi(c)\|_{\Sigma/X_0} \le t. \tag{3.6}$$

Let us construct such an element. Recall previously that $B' \cap B'' \ne \emptyset$ for every two sets $B' = s'\bar{B}_0 + t'\bar{B}_1 + c'$ and $B'' = s''\bar{B}_0 + t''\bar{B}_1 + c''$ of $\mathcal{B}'$. Therefore, by (2.5),

$$c' - c'' \in (s' + s'')\bar{B}_0 + (t' + t'')\bar{B}_1.$$

Since X_0 is closed in Σ, this implies

$$c' - c'' \in X_0 + (t' + t'')B_\Sigma,$$

which is equivalent to the inequality

$$\|\pi(c') - \pi(c'')\|_{\Sigma/X_0} \le t' + t''. \tag{3.7}$$

Let us consider now the sequence of sets $B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i \in \mathcal{B}'$ satisfying the equalities (3.1). Then

$$\|\pi(c_i) - \pi(c_j)\|_{\Sigma/X_0} \le t_i + t_j \to 0, \quad \text{as } i,j \to \infty.$$

Thus the sequence $\{\pi(c_i)\}$ is fundamental and consequently converges to an element $z \in \Sigma/X_0$. Then, by (3.7),

$$\begin{aligned}\|z - \pi(c)\|_{\Sigma/X_0} &\le \|z - \pi(c_i)\|_{\Sigma/X_0} + \|\pi(c_i) - \pi(c)\|_{\Sigma/X_0}\\ &\le \|z - \pi(c_i)\|_{\Sigma/X_0} + t_i + t.\end{aligned}$$

Directing now i to ∞ we obtain the desired inequality (3.6). The lemma is proved. ∎

LEMMA 3.7: *If X_0 is a closed subspace of $X_0 + X_1$, then the family $\mathcal{B}'$ possesses Property* 3.1.

Proof: Let $\delta = \delta(\varepsilon)$ be a positive number which we will indicate below.

Let us show first that for any set $B = s\bar{B}_0 + t\bar{B}_1 + c$ there exists a decomposition

$$c = a_{X_0} + x_B + y_B, \tag{3.8}$$

where a_{X_0} is the element constructed in Lemma 3.6, $x_B \in X_0$, $y_B \in X_1$ and

$$y_B \in (1+\delta)tB_1. \tag{3.9}$$

This is immediate in the case $t = 0$, since then we can set $y_B = 0$, $x_B = c - a_{X_0}$. So we may suppose that $t > 0$. Then, by Lemma 3.6, there exists $b \in X_0$ such that

$$\|c - a_{X_0} - b\|_\Sigma \le t + \frac{\delta}{2}t = \Big(1 + \frac{\delta}{2}\Big)t.$$

Consequently, by definition (2.1), there exist elements $x_0 \in X_0$ and $x_1 \in X_1$ such that

$$c - a_{X_0} - b = x_0 + x_1,$$

$$\|x_0\|_0 + \|x_1\|_1 \le \Big(1 + \frac{\delta}{2}\Big)t + \frac{\delta}{2}t = (1+\delta)t.$$

It remains to set $x_B = b + x_0$, $y_B = x_1$, and (3.8) and (3.9) are proved.

Let $\{B^{(i)}, i = 1, 2, \ldots\}$ be a subfamily of $\mathcal{B}'$, satisfying the equalities (3.1). We can assume that the number u in (3.1) is positive. If $u = 0$ then 0 is a limit point of $\mathcal{T}(\mathcal{B}')$ which, by Lemma 3.4, completes the proof.

So we may assume that $u > 0$. In that case there exists a positive integer i_0 such that $s_{i_0} \le (1+\delta)u$ and $t_{i_0} \le \delta u$. For the sake of brevity we set

$$\tilde{c} = c_{i_0}, \quad \tilde{t} = t_{i_0}, \quad \tilde{s} = s_{i_0}, \quad \tilde{x} = x_{B^{(i_0)}}, \quad \tilde{y} = y_{B^{(i_0)}}.$$

Then, by definition, the set $\tilde{B} = \tilde{s}\bar{B}_0 + \tilde{t}\bar{B}_1 + \tilde{c}$ belongs to $\mathcal{B}'$ and

$$\tilde{s} \le (1+\delta)u, \quad \tilde{t} \le \delta u. \tag{3.10}$$

In addition, by (3.8) and (3.9), $\tilde{c} = a_{X_0} + \tilde{x} + \tilde{y}$, $\tilde{x} \in X_0$, $\tilde{y} \in X_1$ and

$$\tilde{y} \in (1+\delta)\tilde{t}B_1. \tag{3.11}$$

Finally, let us put

$$\bar{c} = \tilde{c} - \tilde{y} (= a_{X_0} + \tilde{x}) \tag{3.12}$$

and show that for some sufficiently small $\delta = \delta(\varepsilon)$, Property 3.1 holds.

So, let $B = s\bar{B}_0 + t\bar{B}_1 + c \in \mathcal{B}'$. We shall consider the following two cases:

(a) $\tilde{t} \le \frac{1}{3}\varepsilon t$.

Since $B \cap \tilde{B} \ne \emptyset$, by (2.5),

$$c - \tilde{c} \in (\tilde{s} + s)\bar{B}_0 + (\tilde{t} + t)\bar{B}_1.$$

On the other hand, by (3.11) and (3.12),

$$\tilde{c} - \bar{c} = \tilde{y} \in (1+\delta)\tilde{t}B_1.$$

Therefore

$$\begin{aligned} c - \bar{c} &\in (\tilde{s} + s)\bar{B}_0 + (\tilde{t} + t)\bar{B}_1 + (1+\delta)\tilde{t}B_1 \\ &\subset (\tilde{s} + s)\bar{B}_0 + (t + (2+\delta)\tilde{t})\bar{B}_1. \end{aligned} \tag{3.13}$$

Together with (3.10) and the inequality $\tilde{t} \le \varepsilon/3$ this gives

$$c - \bar{c} \in \{(1+\delta)u + s\}\bar{B}_0 + \{1 + (2+\delta)\varepsilon/3\}t\bar{B}_1.$$

From this and (2.5) we have

$$\{(1+\delta)u\bar{B}_0 + \bar{c}\} \cap (1 + \tfrac{1}{3}\varepsilon(2+\delta)) \circ B \neq \emptyset,$$

which in case $\delta \leq \min(1, \varepsilon)$ gives the desired Property 3.1.

(b) $\frac{1}{3}\epsilon t \leq \tilde{t}$. By (3.11),

$$c - \bar{c} = (c - \bar{c} - y_B) + y_B \in (c - \bar{c} - y_B) + (1+\delta)\tilde{t}B_1. \tag{3.14}$$

On the other hand, by (3.13), (3.9), (3.10) and (b), we obtain

$$\begin{aligned} c - \bar{c} - y_B &\in (s + \tilde{s})\bar{B}_0 + (t + (2+\delta)\tilde{t})\bar{B}_1 + (1+\delta)tB_1 \\ &\subset (s+\tilde{s})\bar{B}_0 + (2+\delta)\big(1 + 3/\varepsilon\big)\tilde{t}\bar{B}_1. \end{aligned} \tag{3.15}$$

Furthermore, in view of (3.8) and (3.12) we have

$$c - \bar{c} - y_B = (c - a_{X_0} - y_B) - (\bar{c} - a_{X_0}) = x_B + \tilde{x} \in X_0. \tag{3.16}$$

Recall that X_0 is a closed subspace of $X_0 + X_1$ and consequently $\bar{B}_0 \subset X_0$. Moreover, by the Banach theorem, the norm $\|\cdot\|_0$ and the norm, generated on X_0 by the (X_0+X_1)-norm, are equivalent. It means that for some λ, $\lambda \geq 1$,

$$B_\Sigma \cap X_0 \subset \lambda B_0. \tag{3.17}$$

Now, by (3.15) and (3.16), we get

$$c - \bar{c} - y_B \in \big\{(s+\tilde{s})\bar{B}_0 + (2+\delta)\big(1+3/\varepsilon\big)\tilde{t}\bar{B}_1\big\} \cap X_0.$$

In addition, by (3.17),

$$\bar{B}_1 \cap X_0 \subset B_\Sigma \cap X_0 \subset \lambda B_0$$

which, together with (3.10), gives

$$c - \bar{x} - y_B \in ((1+\delta)u + s)\bar{B}_0 + \lambda(2+\delta)\big(1+3/\varepsilon\big)\delta u\bar{B}_0 = ((1+\alpha)u + s)\bar{B}_0.$$

Here we set

$$\alpha = \alpha(\varepsilon, \delta, \lambda) = \delta\Big(1 + \lambda(2+\delta)\big(1+3/\varepsilon\big)\Big).$$

Now from this and (3.14) we have

$$c - \bar{c} \in ((1+\alpha)u + s)\bar{B}_0 + (1+\delta)tB_1 \subset ((1+\alpha)u + (1+\delta)s)\bar{B}_0 + (1+\delta)t\bar{B}_1.$$

According to (2.5), this means that

$$(1+\alpha) \circ C \cap (1+\delta) \circ B \neq \emptyset,$$

where $C = u\bar{B}_0 + \bar{c}$.

It remains to note that for $\delta = \min(1, \varepsilon^2/9\lambda)$ the inequalities $\alpha \leq \varepsilon$, $\delta \leq \min(1, \varepsilon)$ hold. So, both in the case (a) and in the case (b), Property 3.1 holds and the lemma follows. ∎

4. Proof of Theorem 1.3 and Theorem 1.5

Proof of Theorem 1.3: If condition (i) of the theorem holds, then the desired conclusion follows immediately from Proposition 2.1. So we turn to proving the theorem in the cases where condition (ii) or (iii) holds.

If the set $\mathcal{T}(\mathcal{B}')$ (see (1.5)) has no limit points lying on the coordinate axes, then we can again use Proposition 2.1 to complete the proof in these two cases. So we can suppose that $\mathcal{T}(\mathcal{B}')$ possesses such limit points lying, for example, on the s-axis. We let $s(\mathcal{B}') = (u, 0)$ denote the limit point which has the *minimal first coordinate* among these points. Then for given $\varepsilon > 0$ by Lemma 3.3 (in case condition (ii) of Theorem 1.3 holds) and Lemmas 3.5 and 3.7 (in case condition (iii) holds), there exists a point $\bar{c} \in X_0 + X_1$ such that

$$(1+\varepsilon) \circ C \cap (1+\varepsilon) \circ B \neq \emptyset \quad \text{for all } B \in \mathcal{B}'.$$

Here $C = u\bar{B}_0 + \bar{c}$.

Thus every two sets of the family

$$\mathcal{B}'' = \{(1+\varepsilon) \circ B : B \in \mathcal{B}'\} \cup \{(1+\varepsilon) \circ C\}$$

have a non-empty intersection. Moreover

$$s(\mathcal{B}'') = (1+\varepsilon)s(\mathcal{B}') \in \mathcal{T}(\mathcal{B}'').$$

The same argument which we just gave for the family $\mathcal{B}'$ and the s-axis can now be applied to the family $\mathcal{B}''$ and the t-axis to show that the family

$$(1+\varepsilon) \circ \mathcal{B}'' = \{(1+\varepsilon) \circ B : B \in \mathcal{B}''\}$$

can be extended to a family $\mathcal{B}''' \subset \mathcal{B}(\vec{X})$, satisfying the condition of Proposition 2.1. By this proposition,

$$\bigcap_{B\in\mathcal{B}'''} \gamma \circ B \neq \emptyset$$

where $\gamma = 7 + 4\sqrt{2} + \varepsilon$. Since $\mathcal{B}' \subset (1+\varepsilon) \circ \mathcal{B}'' \subset (1+\varepsilon)^2 \circ \mathcal{B}'''$, we get

$$\bigcap_{B\in\mathcal{B}'} \gamma(1+\varepsilon)^2 \circ B \neq \emptyset$$

which completes the proof. ∎

Proof of Theorem 1.5, part (i): We will need the following

LEMMA 4.1: *Let $a = (u, 0)$, $u \geq 0$ be a limit point of $\mathcal{T}(\mathcal{B}')$. If the space $\Sigma = X_0 + X_1$ is strongly complete with respect to X_0 then for any $\varepsilon > 0$ there exists an element $\bar{c} \in X_0 + X_1$ such that*

$$(1+\varepsilon) \circ C \cap (1+\varepsilon) \circ B \neq \emptyset \quad \text{for all } B \in \mathcal{B}'. \tag{4.1}$$

Here

$$C = (2\lambda_\Sigma(X_0) + 1)u\bar{B}_0 + \bar{c}. \tag{4.2}$$

Proof: Recall that $\lambda_X(Y)$ denotes the constant introduced in Definition 1.4. In case $u = 0$ the point $a = 0$ will be a limit point of $\mathcal{T}(\mathcal{B}')$ and, by Lemma 3.4, the property (4.1) holds for some $\bar{c} \in X_0 + X_1$. Thus for the remainder of this proof we may assume that $u > 0$.

Since $(u, 0)$ is a limit point of $\mathcal{T}(\mathcal{B}')$, there exists a family

$$\{B^{(i)} = s_i\bar{B}_0 + t_i\bar{B}_1 + c_i, i = 1, 2\ldots\} \subset \mathcal{B}'$$

satisfying the condition (3.1). Without loss of generality we can assume that

$$s_i \leq (1+\delta)u$$

where $\delta = \frac{1}{8}\min(1, \varepsilon)$. Then, according to (3.2),

$$c_i - c_j \in (s_i + s_j)\bar{B}_0 + (t_i + t_j)\bar{B}_1 \subset 2(1+\delta)u\bar{B}_0 + (t_i + t_j)B_\Sigma.$$

Consequently,

$$\operatorname{dist}_\Sigma(c_i - c_j, 2(1+\delta)u\bar{B}_0) = 2(1+\delta)u \cdot \operatorname{dist}_\Sigma(c_i - c_j, B_0) \leq t_i + t_j.$$

Since $t_i \to 0$ as $i \to \infty$ (see (3.1)), we have

$$\mathrm{dist}_\Sigma(\tilde{c_i} - \tilde{c_j}, B_0) \to 0, \quad i, j \to \infty$$

where we put $\tilde{c_i} = c_i/(2(1+\delta)u)$. By the assumption, the space $X_0 + X_1$ is strongly complete with respect to X_0 and therefore (see Definition 1.4) there exists an element $\tilde{c} \in X_0 + X_1$ such that

$$\mathrm{dist}_\Sigma(\tilde{c_i} - \tilde{c}, \lambda_0 B_0) \to 0, \quad i \to \infty. \tag{4.3}$$

Here, for the sake of brevity, we set $\lambda_0 = \lambda_\Sigma(X_0) + \delta$. Finally, let us set

$$\bar{c} = 2(1+\delta)u\tilde{c}.$$

Then by (4.3),

$$d_i = \mathrm{dist}_\Sigma(c_i - \bar{c}, 2\lambda_0(1+\delta)uB_0) \to 0, \quad i \to \infty. \tag{4.4}$$

Let us show now that the set C, defined by (4.2), satisfies (4.1). Indeed, by (4.4) and (2.2),

$$\begin{aligned} c_i - \bar{c} &\in 2\lambda_0(1+\delta)uB_0 + (1+\delta)d_i B_\Sigma \\ &\subset 2\lambda_0(1+\delta)uB_0 + 2(1+\delta)^2 d_i(B_0 + B_1). \end{aligned} \tag{4.5}$$

Let now $B = s\bar{B}_0 + t\bar{B}_1 + c$ be an arbitrary set of $\mathcal{B}'$. Since $B \cap B^{(i)} \neq \emptyset$ and $s_i \le (1+\delta)u$ for all i,

$$c - c_i \in (s + s_i)\bar{B}_0 + (t + t_i)\bar{B}_1 \subset (s + (1+\delta)u)\bar{B}_0 + (t + t_i)\bar{B}_1.$$

Together with (4.5) this gives

$$\begin{aligned} c - \bar{c} &\in \{(2\lambda_0 + 1)(1+\delta)u + s + 2(1+\delta)^2 d_i\}\bar{B}_0 + (t + t_i + 2(1+\delta)^2 d_i)\bar{B}_1 \\ &\subset (1+\delta)\{(2\lambda_0+1)u + s\}\bar{B}_0 + t\bar{B}_1 + \delta_i B_\Sigma, \end{aligned} \tag{4.6}$$

where $\delta_i = 4(1+\delta)^2 d_i + t_i$. Since t_i and $d_i \to 0$ as $i \to \infty$, the sequence $\delta_i \to 0$ as well. Setting now

$$\tilde{B} = ((1+\delta)(2\lambda_0 + 1)u + s)\bar{B}_0 + t\bar{B}_1,$$

by (4.6), (2.3) and (2.4), we get

$$c - \bar{c} \in \bigcap_i (\tilde{B} + \delta_i B_\Sigma) = \mathrm{cl}_\Sigma(\tilde{B}) \subset (1+\delta) \circ \tilde{B}.$$

From this and the inequality $\delta \leq \min(1, \varepsilon/8)$ we have

$$\begin{aligned} c - \bar{c} &\in \{(1+\delta)^2(2\lambda_0 + 1)u + (1+\delta)s\}\bar{B}_0 + (1+\delta)t\bar{B}_1 \\ &\subset \{(1+\varepsilon)(2\lambda_0+1)u + (1+\varepsilon)s\}\bar{B}_0 + (1+\varepsilon)t\bar{B}_1, \end{aligned}$$

which together with (2.5) gives the desired property (4.1). ∎

We are now in a position to complete the proof of part (i) of Theorem 1.5. So, suppose that $X_0 + X_1$ is strongly complete both with respect X_0 and with respect to X_1 and that the family of sets $\mathcal{B}'$ satisfies the conditions of the theorem. Note at once that if $\mathcal{T}(\mathcal{B}')$ *has no* limit points lying on the coordinate axes, then the statement of the theorem follows from Proposition 2.1.

Assume now that $\mathcal{T}(\mathcal{B}')$ possesses such limit points. Applying Lemma 4.1, first to X_0 (and the s-axis) and then a second time to X_1 (and the t-axis), we can extend the family $\mathcal{B}'$ to a family $\tilde{\mathcal{B}} \subset \mathcal{B}(\vec{X})$ such that:

(i) every two sets of the family

$$\mu \circ \tilde{\mathcal{B}} = \{\mu \circ B : B \in \tilde{\mathcal{B}}\}$$

where

$$\mu = (1+\varepsilon)^2(2\max(\lambda_\Sigma(X_0), \lambda_\Sigma(X_1)) + 1)$$

have a common point;

(ii) the points $s(\mu \circ \tilde{\mathcal{B}})$ and $t(\mu \circ \tilde{\mathcal{B}})$ belong to $\mathcal{T}(\mu \circ \tilde{\mathcal{B}})$.

It remains to apply to the family $\mu \circ \tilde{\mathcal{B}}$ the result of Proposition 2.1 and the proof is complete. ∎

Proof of Theorem 1.5, part (ii): Show that under the conditions of the theorem the space $X_0 + X_1$ is strongly complete with respect to X_0 (the same proof is valid for X_1). Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $X_0 + X_1$ such that

$$d_{ij} = \mathrm{dist}_\Sigma(x_i - x_j, B_0) \to 0, \quad i,j \to \infty. \tag{4.7}$$

According to Definition 1.4, we have to establish the existence of an element $\bar{x} \in X_0 + X_1$ such that

$$\mathrm{dist}_\Sigma(x_i - \bar{x}, \lambda B_0) \to 0, \quad i \to \infty.$$

To this end we set

$$d_i = \sup_{j \geq i} d_j.$$

Then by (4.7) and (2.2), for every $i, j,\ i \leq j$,

$$x_i - x_j \in \bar{B}_0 + 2d_{ij} B_\Sigma \subset \bar{B}_0 + 4d_{ij}(B_0 + B_1)$$
$$\subset (1 + 4(d_i + d_j))\bar{B}_0 + 4(d_i + d_j)\bar{B}_1.$$

From this and (2.5) it follows that $B^{(i)} \cap B^{(j)} \neq \emptyset$ where

$$B^{(i)} = (\tfrac{1}{2} + 4d_i)\bar{B}_0 + 4d_i\bar{B}_1 + x_i.$$

Thus every two sets of the family $\mathcal{B}' = \{B^{(i)} : i = 1, 2, \ldots\}$ have a common point. Therefore, by the assumption of the theorem, there exists an element $\bar{x} \in X_0 + X_1$ such that

$$\bar{x} \in \gamma \circ B^{(i)}, \quad i = 1, 2, \ldots.$$

Consequently,

$$x_i - \bar{x} \in \gamma(\tfrac{1}{2} + 4d_i)\bar{B}_0 + 4\gamma d_i \bar{B}_0 \subset \tfrac{1}{2}\gamma\bar{B}_0 + 8\gamma d_i B_\Sigma.$$

By (4.7), $d_i \to 0$ as $i \to \infty$. Thus

$$\operatorname{dist}_\Sigma(x_i - \bar{x}, \tfrac{1}{2}\gamma B_0) \leq 8\gamma d_i \to 0, \quad i \to \infty$$

and part (ii) of Theorem 1.5 follows. ∎

5. Example 5.1 and proof of Theorem 1.2

Given a positive integer N define sequences $\{s_m\}, \{t_m\}, \{r_m\}$ by the formulas

$$s_m = N^{m^2}, \quad t_m = s_m^{-1}, \quad r_m = s_m^{-2}, \quad m = 0, 1, \ldots, N$$

and set

$$a_k = s_{k-1} r_k + t_{k-1}, \quad k = 1, 2, \ldots, N.$$

Let X_0 and X_1 denote the space $\mathbf{R}^N$ equipped with the norms $\|x\|_0 = \max_{i=1,\ldots,N} |x_i|$ and $\|x\|_1 = \max_{i=1,\ldots,N} |x_i|/r_i$, respectively. We shall choose X_2 to be the linear subspace of $\mathbf{R}^N$,

$$X_2 = \left\{x \in \mathbf{R}^N \colon \sum_{i=1}^{N} \frac{x_i}{a_i} = 0\right\},$$

with the norm $\|x\|_2 = \max_{i=1,\dots,N} |x_i|/a_i$. Finally, define the family $\mathcal{B}' = \{B^{(m)}\colon m = 0, 1, \dots, N\} \subset \mathcal{B}(\vec{X})$, where $\vec{X} = (X_0, X_1, X_2)$, setting

$$\begin{aligned} B^{(0)} &= B_2 + a_1\vec{e}_1 \\ &= \Big\{x \in \mathbf{R}^N\colon \sum_{i=1}^{N} \frac{x_i}{a_i} = 1, \Big|\frac{x_1}{a_1} - 1\Big| \le 1, |x_i| \le a_i, i = 2, \dots, N\Big\} \end{aligned}$$

and

$$B^{(m)} = s_m B_0 + t_m B_1 = \{x \in \mathbf{R}^N : |x_i| \le s_m r_i + t_m, i = 1, \dots, N\}.$$

Here B_i is the unit ball of X_i and $\vec{e}_k$ denotes the k-th vector of the standard basis of $\mathbf{R}^N$. It is not difficult to check that for every $k = 0, \dots, N$

$$\bigcap_{m \ne k} B^{(m)} \ne \emptyset,$$

but, nevertheless,

$$\bigcap_{B \in \mathcal{B}'} \gamma \circ B = \emptyset \qquad \text{for all } \gamma \in (0, N/4).$$

In particular, this shows that the number $m = m(3)$ introduced in Remark 1.7 satisfies

$$m(3) \ge N/4 \to \infty \qquad \text{as } N \to \infty. \qquad \blacksquare$$

Proof of Theorem 1.2: Let $K_\infty(t; x : X)$ denote the K_∞-functional of an element $x \in X_0 + X_1$ with respect to the couple $\vec{X} = (X_0, X_1)$, i.e.,

$$K_\infty(t; x : \vec{X}) = \inf_{x = x_0 + x_1} \{\max(\|x_0\|_0, t\|x_1\|_1\}. \tag{5.1}$$

Recall (see, e.g., [BL]) that

$$K_\infty(t; x : \vec{X}) = \sup_{s>0} \frac{t}{t+s} K_\infty(s; x : \vec{X}) \tag{5.2}$$

and

$$K(t; x\colon \vec{X}) = \inf_{s>0} \Big(1 + \frac{t}{s} K_\infty(s; x : \vec{X})\Big). \tag{5.3}$$

Thus by the assumption (1.3) and (5.2),

$$K_\infty(t; x_\alpha - x_\beta \colon \vec{X}) \le \psi_\alpha(t) + \psi_\beta(t), \quad t > 0 \tag{5.4}$$

where

$$\psi_\alpha = \sup_{s>0} \frac{t}{t+s} \varphi_\alpha(s). \tag{5.5}$$

Next, for any $\varepsilon > 0$, by (5.1) and (5.4), we have

$$x_\alpha - x_\beta \in (1+\varepsilon)(\psi_\alpha(t) + \psi_\beta(t)) \left(B_0 + \frac{1}{t} B_1 \right), \quad t > 0. \tag{5.6}$$

Let us show that (5.6) implies

$$x_\alpha - x_\beta \in 2(1+\varepsilon)(\psi_\alpha(t) + \psi_\beta(t)) B_0 + \left(\frac{\psi_\alpha(t)}{t} + \frac{\psi_\beta(s)}{s} \right) B_1 \tag{5.7}$$

for all $s, t > 0$. Indeed, suppose that $0 < s \le t$ and set

$$u = (\psi_\alpha(s) + \psi_\beta(t)) / \left(\frac{\psi_\alpha(s)}{s} + \frac{\psi_\beta(t)}{t} \right).$$

Obviously, $0 < s \le u \le t$. Moreover, by (5.5), the functions $\psi_\alpha(t)$ and $t/\psi_\alpha(t)$ are both non-decreasing. Therefore

$$\psi_\alpha(u) + \psi_\beta(u) \le \psi_\alpha(s) + 2\psi_\beta(t),$$
$$\frac{\psi_\alpha(u)}{u} + \frac{\psi_\beta(u)}{u} \le \frac{2\psi_\alpha(s)}{s} + \frac{\psi_\beta(t)}{t}.$$

It remains to set $t = u$ in (5.6) and the desired relation (5.7) follows.

Set now

$$B^{(\alpha)}(t) = 2(1+\varepsilon)\psi_\alpha(t) \left\{ \bar{B}_0 + \frac{1}{t} \bar{B}_1 \right\} + x_\alpha, \quad t > 0, \quad \alpha \in \mathcal{A}.$$

Then by (5.7) and (2.5), every two sets of the family

$$\mathcal{B}' = \{ B^{(\alpha)}(t) : t > 0, \alpha \in \mathcal{A} \}$$

have a common point. Moreover, by the assumptions (ii) and (iii) of the theorem, the family $\mathcal{B}'$ satisfies the conditions (ii) and (iii) of Theorem 1.3. According to this theorem, there exists an element $\bar{x} \in X_0 + X_1$ such that for all $\alpha \in \mathcal{A}$ and $t > 0$,

$$\bar{x} \in \gamma \circ B^{(\alpha)}(t).$$

Here γ is the constant defined in (1.2). Thus

$$x_\alpha - \bar{x} \in 2(1+\varepsilon)\gamma\psi_\alpha(t)\left\{\bar{B}_0 + \frac{1}{t}\bar{B}_1\right\} \subset 2(1+\varepsilon)^2\gamma\psi_\alpha(t)\left\{B_0 + \frac{1}{t}B_1\right\}.$$

From this and definition (5.1) we get

$$K_\infty(t; x_\alpha - \bar{x} : \vec{X}) \leq 2(1+\varepsilon)^2\gamma\psi_\alpha(t)$$

for all $t > 0$ and $\alpha \in \mathcal{A}$. Now by (5.3),

$$K(t; x_\alpha - \bar{x} \colon \vec{X}) \leq 2(1+\varepsilon)^2\gamma \inf_{s>0}(1 + t/s)\psi_\alpha(s).$$

Since φ_α is a concave and non-decreasing function on $\mathbf{R}_+$, the latter infimum is equal to $\varphi_\alpha(t)$. This completes the proof of Theorem 1.2. ∎

References

[BK1] Yu. Brudnyi and N. Krugljak, *Real interpolation functors*, Doklady Akademii Nauk SSSR **256** (1981), 14–17; English transl.: Soviet Mathematics Doklady **23** (1981), 5–8.

[BK2] Yu. Brudnyi and N. Krugljak, *Interpolation Functors and Interpolation Spaces, I*, North-Holland Math. Library **47**, North-Holland, Amsterdam, 1991.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, 1976.

[BS] Yu. Brudnyi and A. Shteinberg, *Calderón couples of Lipschitz spaces*, Journal of Functional Analysis **191** (1995), 459–498.

[BSh] C. Bennet and R. Sharpley, *K-divisibility and a theorem of Lorentz and Shimogaki*, Proceedings of the American Mathematical Society **96** (1986), 585–592.

[C] M. Cwikel, *K-divisibility of the K-functional and Calderón couples*, Arkiv för Mathematik **22** (1984), 39–162.

[CJM] M. Cwikel, B. Jawerth and M. Milman, *On the fundamental lemma of interpolation theory*, Journal of Approximation Theory **60** (1990), 70–82.

[DGK] L. Danzer, B. Grünbaum and V. Klee, *Helly's Theorem and its relatives*, in *Convexity*, Proceedings of Symposia in Pure Mathematics **7** (1963), 101–180.

[K] N. Krugljak, *On the reiteration property of $\vec{X}_{\phi,q}$ spaces*, Mathematica Scandinavica **73** (1993), 65–80.

[K1] N. Krugljak, *Private communication.*

[M] M. Milman, *Extrapolation and Optimal Decomposition*, Lecture Notes in Mathematics **1580**, Springer-Verlag, Berlin, 1994.

[N1] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Annali di Matematica Pura ed Applicata **132** (1982), 291–330.

[N2] P. Nilsson, *Interpolation of Calderón pairs and Ovchinnikov pairs*, Annali di Matematica Pura ed Applicata **134** (1983), 201–232.

[S] P. Shvartsman, *K-divisibility and interpolation of weighted Lipschitz spaces*, in preparation.